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We determine the behaviour of a time-varying fine structure 'constant'  $\alpha(t)$  during the early and late phases of universes dominated by the kinetic energy of changing  $\alpha(t)$ , radiation, dust, curvature, and  $\lambda$ , respectively. We show that after leaving an initial vacuum-dominated phase during which  $\alpha$  increases,  $\alpha$  remains constant in universes like our own during the radiation era, and then increases slowly, proportional to a logarithm of cosmic time, during the dust era. If the universe becomes dominated by negative curvature or a positive cosmological constant then  $\alpha$  tends rapidly to a constant value. The effect of an early period of de Sitter or power-law inflation is to drive  $\alpha$  to a constant value. Various cosmological consequences of these results are discussed with reference to recent observational studies of the value of  $\alpha$  from quasar absorption spectra and to the existence of life in expanding universes.

## I. INTRODUCTION

One of the problems that cosmologists have faced in their attempts to assess the astronomical consequences of a time variation in the fine structure constants,  $\alpha$ , has been the absence of an exact theory describing cosmological models in the presence of varying  $\alpha$ . Until recently, it has not been possible to analyse the behaviour of varying- $\alpha$  cosmologies in the same self-consistent way that one can explore universes with varying  $G$  using the Brans-Dicke or more general scalar-tensor theories of gravity. However, we have recently extended the generalisation of Maxwell's equations developed by Bekenstein so that this can be done self-consistently. In a recent letter [1] we reported numerical studies of the cosmological evolution of varying- $\alpha$  cosmologies with zero curvature, non-zero cosmological constant, and matter density matching observations. They reveal important properties of varying- $\alpha$  cosmologies that are shared by other theories in which 'constants' vary via the propagation of a causal scalar field obeying 2nd-order differential equations. Their structure can be compared with that of varying speed of light (VSL) theories developed in refs. [2–8] and with Kaluza-Klein like theories in which constants like  $\alpha$  vary at the same rate as the mean size of any extra dimensions of space [25].

Recent observations motivate the formulation and detailed investigation of varying- $\alpha$  cosmological theories. The new observational many-multiplet technique of Webb et al, [9], [10], exploits the extra sensitivity gained by studying relativistic transitions to different ground states using absorption lines in quasar (QSO) spectra at medium redshift. It has provided the first evidence that the fine structure constant might change with cosmological time [9–11]. The trend of these results is that the value of  $\alpha$  was lower in the past, with

$\Delta\alpha/\alpha = -0.72 \pm 0.18 \times 10^{-5}$  for  $z \approx 0.5 - 3.5$ . Other investigations have claimed preferred non-zero values of  $\Delta\alpha < 0$  to best fit the cosmic microwave background (CMB) and Big Bang Nucleosynthesis (BBN) data at  $z \approx 10^3$  and  $z \approx 10^{10}$  respectively [12,13], but these need to be much larger than those needed to reconcile the observations of [9–11].

In this paper we present a detailed analytic and numerical study of the behaviour of the cosmological solutions of the varying theory presented in [1]. We shall confine our attention to universes containing dust and radiation but analyse the effects of negative spatial curvature and a positive cosmological constant. Extensions to general perfect-fluid cosmologies can easily be made if required.

## II. A SIMPLE VARYING-ALPHA THEORY

The idea that the charge on the electron, or the fine structure constant, might vary in cosmological time was proposed in 1948 by Teller, [14], who suggested that  $\alpha \propto (\ln t)^{-1}$  was implied by Dirac's proposal that  $G \propto t^{-1}$  and the numerical coincidence that  $\alpha^{-1} \sim \ln(hc/Gm_{pr}^2)$ , where  $m_{pr}$  is the proton mass. Later, in 1967, Gamow [15] suggested  $\alpha \propto t$  as an alternative to Dirac's time-variation of the gravitation constant,  $G$ , as a solution of the large numbers coincidences problem but in 1963 Stanyukovich had also considered varying  $\alpha$ , [16], in this context. It had the advantage of not producing a terrestrial surface temperature above 100 degrees centigrade in the pre-Cambrian era when life was known to exist. However, this power-law variation in the recent geological past was soon ruled out by other evidence.

There are a number of possible theories allowing for the variation of the fine structure constant,  $\alpha$ . In the simplest cases one takes  $c$  and  $\hbar$  to be constants and at-

tributes variations in  $\alpha$  to changes in  $e$  or the permittivity of free space (see [3] for a discussion of the meaning of this choice). This is done by letting  $e$  take on the value of a real scalar field which varies in space and time (for more complicated cases, resorting to complex fields undergoing spontaneous symmetry breaking, see the case of fast tracks discussed in [7]). Thus  $e_0 \rightarrow e = e_0 \epsilon(x^\mu)$ , where  $\epsilon$  is a dimensionless scalar field and  $e_0$  is a constant denoting the present value of  $e$ . This operation implies that some well established assumptions, like charge conservation, must give way [23]. Nevertheless, the principles of local gauge invariance and causality are maintained, as is the scale invariance of the  $\epsilon$  field (under a suitable choice of dynamics). In addition there is no conflict with local Lorentz invariance or covariance.

With this set up in mind, the dynamics of our theory is then constructed as follows. Since  $e$  is the electromagnetic coupling, the  $\epsilon$  field couples to the gauge field as  $\epsilon A_\mu$  in the Lagrangian and the gauge transformation which leaves the action invariant is  $\epsilon A_\mu \rightarrow \epsilon A_\mu + \chi_{,\mu}$ , rather than the usual  $A_\mu \rightarrow A_\mu + \chi_{,\mu}$ . The gauge-invariant electromagnetic field tensor is therefore

$$F_{\mu\nu} = \frac{1}{\epsilon} ((\epsilon A_\nu)_{,\mu} - (\epsilon A_\mu)_{,\nu}), \quad (1)$$

which reduces to the usual form when  $\epsilon$  is constant. The electromagnetic part of the action is still

$$S_{em} = - \int d^4x \sqrt{-g} F^{\mu\nu} F_{\mu\nu}. \quad (2)$$

and the dynamics of the  $\epsilon$  field are controlled by the kinetic term

$$S_\epsilon = -\frac{1}{2} \frac{\hbar c}{l^2} \int d^4x \sqrt{-g} \frac{\epsilon_{,\mu} \epsilon^{,\mu}}{\epsilon^2}, \quad (3)$$

as in dilaton theories. Here,  $l$  is the characteristic length scale of the theory, introduced for dimensional reasons. This constant length scale gives the scale down to which the electric field around a point charge is accurately Coulombic. The corresponding energy scale,  $\hbar c/l$ , has to lie between a few tens of  $MeV$  and Planck scale,  $\sim 10^{19} GeV$  to avoid conflict with experiment.

Our generalisation of the scalar theory proposed by Bekenstein [17] described in ref. [1] includes the gravitational effects of  $\psi$  and gives the field equations:

$$G_{\mu\nu} = 8\pi G (T_{\mu\nu}^{matter} + T_{\mu\nu}^\psi + T_{\mu\nu}^{em} e^{-2\psi}). \quad (4)$$

The stress tensor of the  $\psi$  field is derived from the Lagrangian  $\mathcal{L}_\psi = -\frac{\omega}{2} \partial_\mu \psi \partial^\mu \psi$  and the  $\psi$  field obeys the equation of motion

$$\square \psi = -\frac{2}{\omega} e^{-2\psi} \mathcal{L}_{em} \quad (5)$$

where we have defined the coupling constant  $\omega = (\hbar c)/l^2$ . This constant is of order  $\sim 1$  if, as in [1], the energy scale

is similar to Planck scale. In order to make quantitative predictions we need to know how much of the non-relativistic matter contributes to the RHS of Eqn. (5). This is parametrised by  $\zeta \equiv \rho_{em}/\rho_{matter}$ . In [17],  $\zeta$  was estimated to be around 1%. However, if we choose to model the proton as a charged shell of radius equal to the estimated proton radius, the fraction would be lower, approximately 0.19%. Also, the value of  $\zeta$  needs to be weighted by the fraction of matter that is non-baryonic, a point ignored in the literature [17,18]. Hence, the total  $\zeta$  depends strongly on the nature of the dark matter. BBN predicts an approximate value for the baryon density of  $\Omega_B \approx 0.0125 h_0^{-2}$ , or  $\Omega_B \approx 0.03\%$  with a Hubble parameter of  $h_0 \approx 0.6$ . Since we believe the total matter density to be  $\Omega_m \approx 0.3$ , this would mean that only about 1/10 of matter is baryonic and couples to changes in  $e$ . Thus, we should assume values for  $\zeta$  ranging from 0.02% to 0.1%, if the cold dark matter is allowed to have little or no electrostatic Coulomb component. If this is not true, then  $\zeta$  could have a much higher value.

We should not confuse this theory with other similar variations. Bekenstein's theory [17] does not take into account the stress energy tensor of the dielectric field in Einstein's equations, and their application to cosmology. Dilaton theories predict a global coupling between the scalar and all other matter fields. As a result they predict variations in other constants of nature, and also a different dynamics to all the matter coupled to electromagnetism. An interesting application of our approach has also recently been made to braneworld cosmology in [19].

### III. THE COSMOLOGICAL EQUATIONS

Assuming a homogeneous and isotropic Friedmann metric with expansion scale factor  $a(t)$  and curvature parameter  $k$  in eqn. (4), we obtain the field equations ( $c \equiv 1$ )

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \left( \rho_m (1 + \zeta \exp[-2\psi]) + \rho_r \exp[-2\psi] + \frac{\omega}{2} \dot{\psi}^2 \right) - \frac{k}{a^2} + \frac{\Lambda}{3}, \quad (6)$$

where  $\Lambda$  is the cosmological constant. For the scalar field we have the propagation equation,

$$\ddot{\psi} + 3H\dot{\psi} = \frac{2}{\omega} \exp[-2\psi] \zeta \rho_m, \quad (7)$$

where  $H \equiv \dot{a}/a$  is the Hubble expansion rate. The conservation equations for the non-interacting radiation, and matter densities are

$$\dot{\rho}_m + 3H\rho_m = 0 \quad (8)$$

$$\dot{\rho}_r + 4H\rho_r = 2\dot{\psi}\rho_r. \quad (9)$$

and so  $\rho_m \propto a^{-3}$  and  $\rho_r e^{-2\psi} \propto a^{-4}$ , respectively. If additional non-interacting perfect fluids satisfying equation of state  $p = (\gamma - 1)\rho$  are added to the universe then they contribute density terms  $\rho \propto a^{-3\gamma}$  to the RHS of eq.(6) as usual. This theory enables the cosmological consequences of varying  $e$ , to be analysed self-consistently rather than by changing the constant value of  $e$  in the standard theory to another constant value, as in the original proposals made in response to the large numbers coincidences (see ref. [20] for a full discussion).

We have been unable to solve these equations in general except for a few special cases. However, as with the Friedmann equation of general relativity, it is possible to determine the overall pattern of cosmological evolution in the presence of matter, radiation, curvature, and positive cosmological constant by matched approximations. We shall consider the form of the solutions to these equations when the universe is successively dominated by the kinetic energy of the scalar field  $\psi$ , pressure-free matter, radiation, negative spatial curvature, and positive cosmological constant. Our analytic expressions are checked by numerical solutions of (6) and (7).

### A. The Dust-dominated era

We consider first the behaviour of dust-filled universes far from the initial singularity. We assume that  $k = 0 = \Lambda = \rho_\gamma$ , so the Friedmann equation (6) reduces to

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \left( \rho_m (1 + \zeta \exp[-2\psi]) + \frac{\omega}{2} \dot{\psi}^2 \right), \quad (10)$$

and seek a self-consistent approximate solution in which the scale factor behaves as

$$a = t^{2/3} \quad (11)$$

$$\frac{d}{dt}(\dot{\psi} a^3) = N \exp[-2\psi] \quad (12)$$

where

$$N \equiv \frac{2\zeta}{\omega} \rho_m a^3 \quad (13)$$

is a positive constant. If we put

$$x = \ln(t)$$

then (12) becomes

$$\psi'' + \psi' = N \exp[-2\psi] \quad (14)$$

with  $N \geq 0$  and  $' \equiv d/dx$ . This equation has awkward behaviour. For any power-law behaviour of the scale factor other than (11) a simple exact solution of (12) exists. However, the late-time dust solutions are exceptional, reflecting the coupling of the charged matter to the variations in  $\psi$ , and are approximated by the following asymptotic series:

$$\psi = \frac{1}{2} \ln[2Nx] + \sum_{n=1}^{\infty} a_n x^{-n} \quad (15)$$

To see this, substitute this in the evolution eqn. (14) for  $\psi$  then it becomes:

$$-\frac{1}{2x^2} + \sum_{n=1}^{\infty} n(n+1)a_n x^{-n-2} + \frac{1}{2x} - \sum_{n=1}^{\infty} n a_n x^{-n-1} = \frac{1}{2x} \exp[-2 \sum_{n=1}^{\infty} a_n x^{-n}] \quad (16)$$

Now we can pick the  $a_n$  to cancel out all the terms in  $x^{-r}$ ,  $r \geq 2$  on the left-hand side. This requires

$$a_2 = a_1 = -\frac{1}{2}, a_3 = 2a_2, a_4 = 3a_3 = 3 \times 2a_2, \text{etc}$$

hence

$$\sum_{n=1}^{\infty} a_n x^{-n} = -\frac{1}{2} \left\{ \frac{1}{x} + \frac{1}{x^2} + \frac{2}{x^3} + \frac{2 \times 3}{x^4} + \frac{2 \times 3 \times 4}{x^5} + \dots + \frac{(r-1)!}{x^r} + \dots \right\}$$

all that is left of the eqn. (16) is

$$\frac{1}{2x} = \frac{1}{2x} \exp[-2 \sum_{n=1}^{\infty} a_n x^{-n}] \rightarrow \frac{1}{2x}$$

as  $x \rightarrow \infty$ . So, at late times, as  $x = \ln(t)$  becomes large, we have

$$\psi = \frac{1}{2} \ln[2N(\ln(t))] - \frac{1}{2} \left\{ \frac{1}{\ln(t)} + \frac{1}{(\ln(t))^2} + \frac{2}{(\ln(t))^3} + \frac{2 \times 3}{(\ln(t))^4} + \frac{2 \times 3 \times 4}{(\ln(t))^5} + \dots + \frac{(r-1)!}{(\ln(t))^r} + \dots \right\}; \quad (17)$$

also, since  $\alpha = \exp[2\psi]$  we have, as  $t \rightarrow \infty$

$$\alpha = 2N \ln(t) \times \exp\left[-\frac{1}{\ln(t)} - \frac{1}{(\ln(t))^2} - \frac{2}{(\ln(t))^3} - \frac{2 \times 3}{(\ln(t))^4} - \frac{2 \times 3 \times 4}{(\ln(t))^5} - \dots - \frac{(r-1)!}{(\ln(t))^r} - \dots\right]. \quad (18)$$

So, to leading order, we have

$$\alpha \sim 2N \ln(t) \exp\left[-\frac{1}{\ln(t)}\right] \quad (19)$$

The non-analytic  $\exp[1/x]$  behaviour shows why the eqn. (14), despite looking simple, has awkward behaviour. We can simplify the asymptotic series (18) a bit further because we know from the definition of the logarithmic integral function  $li(x) = \int_0^x dt / \ln(t) = \text{Ei}[\ln(x)]$ , that as  $x \rightarrow \infty$

$$li(x) \sim \exp[x] \sum_{n=0}^{\infty} \frac{n!}{x^{n+1}} \quad (20)$$

so the series we have in (17) in  $\{..\}$  brackets is

$$\sum_{r=1}^{\infty} \frac{(r-1)!}{x^r} \sim \exp[-x] \text{li}(\exp[x]) \quad (21)$$

and so asymptotically

$$\psi = \frac{1}{2} \ln[2Nx] - \frac{1}{2} \exp[-x] \text{li}(\exp[x]). \quad (22)$$

Hence, as  $t \rightarrow \infty$ ,

$$\psi = \frac{1}{2} \ln[2N \ln(t)] - \frac{1}{2t} \text{li}(t) = \frac{1}{2} \ln[2N \ln(t)] - \frac{1}{2t} \text{Ei}[\ln(t)] \quad (23)$$

and so asymptotically,

$$\alpha = \exp[2\psi] = 2N \exp[-t^{-1} \text{li}(t)] \ln t. \quad (24)$$

This asymptotic behaviour is confirmed by solving equations (6-9) numerically for  $\rho_m \gg \rho_r, \rho_\psi$ . By using a range of initial values for  $\psi$  we produce the plot in fig (1), in which the asymptotic solution is clearly approached.

We need to check that the original assumption of  $a = t^{2/3}$  in the Friedmann eqn. (6) is self consistent. The relevant terms are

$$\rho_m (1 + \zeta \exp[-2\psi]) + \frac{\omega}{2} \dot{\psi}^2 \quad (25)$$

The  $\exp[-2\psi] = \alpha^{-1}$  falls off as  $t \rightarrow \infty$  so the  $\rho_m (1 + \zeta \exp[-2\psi]) \propto a^{-3}$  term dominates as expected. For the kinetic term  $\dot{\psi}^2$  we have

$$\dot{\psi} = \frac{1}{t} \times O\left(\frac{1}{\ln(t)}\right) \quad (26)$$

and so again the  $\dot{\psi}^2$  term falls off faster than  $t^{-2}$  as  $t \rightarrow \infty$  and the  $a = t^{2/3}$  behaviour is an ever-improving approximation at late times. If we examine the form of the solution (24) we see that  $\alpha$  always *increases* with time as a logarithmic power until it grows sufficiently for the exponential term on the right-hand side of (7) to affect the solution significantly and slow the rate of increase by the series terms. The rate at which  $\alpha$  grows is controlled by the total density of matter in the model, which is directly proportional to the constant  $N$ , defined by eqn. (13). The higher the density of matter (and hence  $N$ ) the faster the growth in  $\alpha$ . However, because of the logarithmic time-variation, the dependence on  $\rho_m, \omega$ , and  $\zeta$  is weak. The self-consistency of the usual  $a = t^{2/3}$  dust evolution for the scale factor leaves the standard cosmological tests unaffected. This is just as one expects for the very variations indicated by the observations of [11].

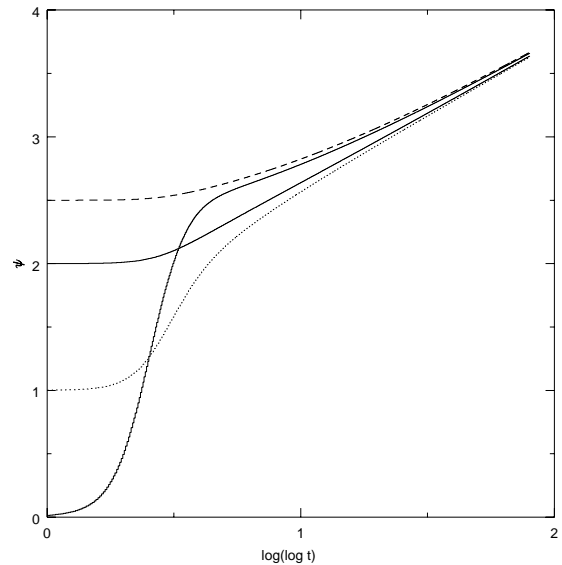


FIG. 1. Numerical solution to the equations in the dust-dominated epoch.  $\psi$  is plotted against  $\log(\log t)$ , with initial conditions  $\psi = 0, 1, 2, 2.5$ . The numerical solution clearly approaches the asymptotic solution in the expected manner. The time is plotted in Planck units of  $10^{-43}$  s.

## B. The Radiation-dominated era

In the radiation era we assume  $k = \Lambda = 0$  and take  $a = t^{1/2}$  as the leading order solution to (6). We must now solve

$$\frac{d}{dt} (\dot{\psi} a^3) = N \exp[-2\psi]. \quad (27)$$

There is a simple particular exact solution

$$\psi = \frac{1}{2} \ln(8N) + \frac{1}{4} \ln(t) \quad (28)$$

Consider a perturbation of this solution by  $f(t)$

$$\psi = \frac{1}{2} \ln(8N) + \frac{1}{4} \ln(t) + f(t)$$

Inserted in eqn. (27) we then get

$$\ddot{f} + \frac{3}{2t} \dot{f} = \frac{1}{8t^2} (\exp[-2f] - 1) \quad (29)$$

Let us first consider the case of a large perturbation,  $\exp(-2f) \ll 1$ . The RHS of (29) then reduces to  $-1/(8t^2)$ , and through a straightforward integration we get

$$\dot{f} = -\frac{1}{4t} + Ct^{-3/2} \quad (30)$$

with  $C$  an arbitrary constant. As  $t$  increases this will approach  $-1/(4t)$  which has the same absolute value and

is opposite in sign to the derivative of the exact solution (28). Thus for values of  $\psi$  much higher than this solution  $\dot{\psi}$  is zero.  $\psi$  will stay constant until the perturbation  $f$  becomes small and  $\psi$  approaches the exact solution (28).

To establish the stability of the exact solution we need to consider small perturbations around it. For small  $f$  we have

$$\ddot{f} + \frac{3}{2t}\dot{f} + \frac{1}{4t^2}f = 0. \quad (31)$$

Hence,

$$f = \frac{1}{t}\{A \sin[\sqrt{3} \ln(t)] + B \cos[\sqrt{3} \ln(t)]\} \quad (32)$$

Thus, we have

$$\begin{aligned} \psi &\rightarrow \frac{1}{2} \ln(8N) + \frac{1}{4} \ln(t) \\ &+ \frac{1}{t}\{A \sin[\sqrt{3} \ln(t)] + B \cos[\sqrt{3} \ln(t)]\} \end{aligned} \quad (33)$$

$$\begin{aligned} \alpha = e^{2\psi} &\rightarrow 8Nt^{1/2} \exp\left[\frac{2}{t}\{A \sin[\sqrt{3} \ln(t)] \right. \\ &\left. + B \cos[\sqrt{3} \ln(t)]\}\right] \rightarrow 8Nt^{1/2} \end{aligned} \quad (34)$$

as  $t \rightarrow \infty$ .

We need to check that the  $\dot{\psi}^2$  term does not dominate as  $t \rightarrow \infty$ . We have

$$\dot{\psi} \sim \frac{1}{4t} + \frac{1}{t^2} \times \text{oscillations} \quad (35)$$

Thus the  $\dot{\psi}^2$  term is the *same order* of  $t$  as the radiation density term if we assume  $a \sim t^{1/2}$ . Also, the matter density term  $\rho_m(1 + \zeta \exp[-2\psi]) \sim \rho_m \exp[-2\psi] \sim a^{-3} \exp[-2\psi] \sim t^{-3/2} \times t^{-1/2} \sim t^{-2}$  is the same order of time variation as the radiation-density term because of the variation in  $\alpha$ . The assumption  $a = t^{1/2}$  is still good asymptotically but there is an algebraic constraint from the Friedmann eqn. (6)

Evaluating the terms in (6), we have

$$\frac{1}{4t^2} = \frac{8\pi G}{3} \left( \frac{M}{t^{3/2}} \left[ 1 + \frac{S}{8Nt^{1/2}} \right] \right) + \frac{\Gamma}{t^2} + \frac{\omega}{32t^2} \quad (36)$$

where  $\rho_m = Ma^{-3}$ ,  $\rho_\gamma \exp[-2\psi] = \Gamma a^{-4}$ ,  $N = 2M\zeta\omega^{-1}$  and we have  $\zeta \sim 0.02\% - 0.1\%$  and probably  $\omega \sim 1$ . So, to  $O(t^{-2})$ , we have the algebraic constraint

$$\frac{1}{4} = \frac{8\pi G}{3} \left[ \frac{3\omega}{32} + \Gamma \right]$$

This generalises the familiar general relativity ( $\omega = 0$ ) radiation universe case where we have  $\Gamma = 3/32\pi G$ .

Again, the asymptotic behaviour in eqns. (33)-(34), and the approach to the exact solution (28), can be confirmed by numerical solutions to eqns.(6) - (9) in the case of radiation domination. The results from runs with initial values for  $\psi = -8, 0, 8$ ,  $\dot{\psi} = 0$  and same value for

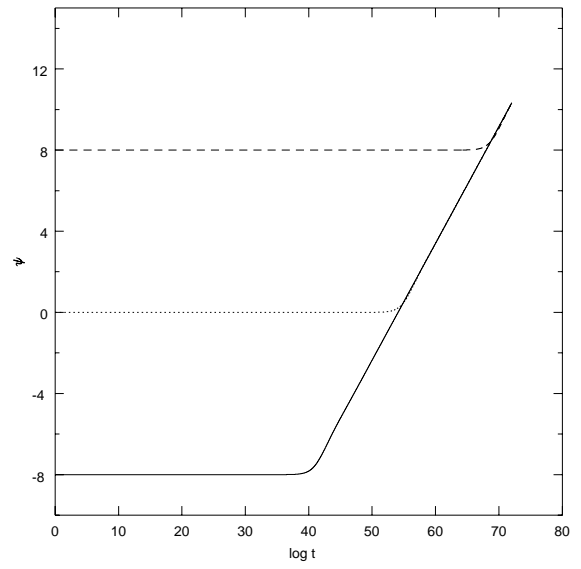


FIG. 2. Numerical solution to the equations in the radiation-dominated epoch given different initial conditions. The particular exact solution is eventually reached in all cases. The time is plotted in units of the Planck time.

$N$ , are shown in fig.(2). The particular solution (28) is clearly an attractor. It is also seen that if the system starts off with values higher than  $1/2 \ln(8N)$ ,  $\psi$  will stay constant until it reaches the value of the solution, as predicted above. In cosmological models containing matter and radiation with densities given by those observed in our universe this is the case, as seen in the computations shown in ref. [1]. Hence, during the radiation era  $\alpha$  remains approximately constant until the dust era begins.

This analysis can easily be extended to other equations of state. If the Friedmann equation contains a perfect fluid with equation of state  $p = (\gamma - 1)\rho$  with  $\gamma \neq 0, 1, 2$  then there is a late time solution of (6) and (7) of the form

$$a = t^{\frac{2}{3\gamma}} \quad (37)$$

$$\psi = \frac{1}{2} \ln\left[\frac{N\gamma^2}{(\gamma-1)(2-\gamma)}\right] + \left(\frac{\gamma-1}{\gamma}\right) \ln(t) \quad (38)$$

which reduces to (28) when  $\gamma = 4/3$ . This solution only exists for fluids with  $1 < \gamma < 2$ .

### C. The Curvature-dominated era

In our earlier study [1] we showed that the evolution of  $\alpha$  stops when the universe becomes dominated by the cosmological constant. This behaviour also occurs when an open universe becomes dominated by negative spatial curvature. In a curvature-dominated era we assume that (6) has the Milne universe solution with

$$a = t. \quad (39)$$

We must now solve eq. (27) again. It has the form

$$\frac{d}{dt}(\dot{\psi}t^3) = N \exp[-2\psi]. \quad (40)$$

We seek a solution of the form

$$\psi = \frac{1}{2} + f(t) \quad (41)$$

Hence, for small  $f$

$$\ddot{f} + \frac{3}{t}\dot{f} + \frac{2N}{t^2}f = 0 \quad (42)$$

Solutions exist with  $f \propto t^n$  and

$$n = -1 \pm \sqrt{1 - 2N} \quad (43)$$

Since  $N > 0$  we see that the real part of  $n$  is always decaying and so

$$\psi \rightarrow \text{const} \quad (44)$$

as  $t \rightarrow \infty$ . Thus, as  $t \rightarrow \infty$  we have

$$\alpha \sim \alpha_\infty \exp[2At^{-1 \pm \sqrt{1-2N}}], \quad (45)$$

where  $\alpha_\infty$  and  $A$  are constants.

Again we need to check that the  $\dot{\psi}^2$  term does not come to dominate. We have  $\dot{\psi}^2 \sim t^{2(n-1)}$  as  $t \rightarrow \infty$  and this always falls faster than  $ka^{-2} \propto t^{-2}$  since  $n \leq 0$ , so our approximation is always good. Thus we have shown that in open Friedmann universes  $\alpha$  rapidly approaches a constant value after the universe becomes curvature dominated. The rate of approach is controlled by the matter density through the constant  $N$  in eq. (45).

This behaviour is again confirmed by numerical solution. Fig.(3) shows how alpha changes through the dust epoch and how the change comes to an end as curvature takes over the expansion.

#### D. The Lambda-dominated era

We can prove what was displayed in the numerical results of [1], and again in fig.(4) for the  $\Lambda$ -dominated era when the value of  $\Lambda$  matches that inferred from recent high redshift supernova observations [27]. At late times we assume the scale factor to take the form

$$a = \exp[\lambda t] \quad (46)$$

where  $\lambda \equiv \sqrt{\frac{\Lambda}{3}}$  and so eqn.(6) becomes

$$\frac{d}{dt}(\dot{\psi}e^{3\lambda t}) = N \exp[-2\psi] \quad (47)$$

Linearising in  $\psi$ , we have

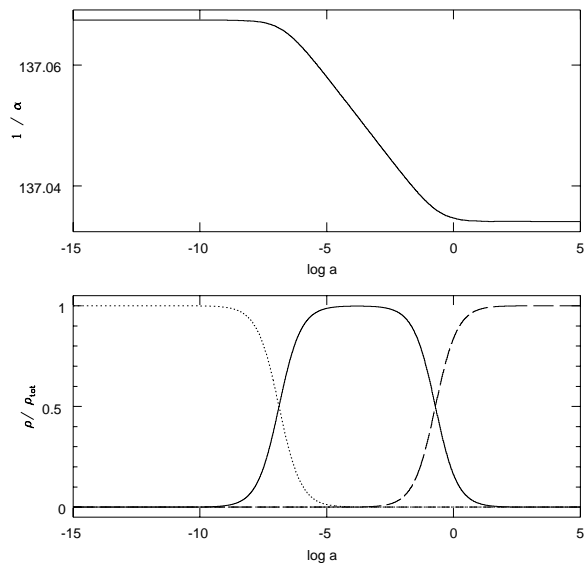


FIG. 3. The top plot shows evolution of  $\alpha$  from radiation domination through matter domination and into curvature domination where the change in  $\alpha$  comes to an end. The lower plot shows radiation (dotted), matter (solid) and curvature (dashed) densities as fractions of the total energy density

$$\ddot{\psi} + 3\lambda\dot{\psi} = N \exp[-3\lambda t]. \quad (48)$$

Hence,

$$\psi = \psi_0 + A \exp[-3\lambda t] - \frac{Nt}{3\lambda} \exp[-3\lambda t] \rightarrow \psi_0 \quad (49)$$

as  $t \rightarrow \infty$ , where  $A, \psi_0$  are arbitrary constants. Thus  $\alpha$  approaches a constant with double-exponential rapidity during a  $\Lambda$ -dominated phase of the universe. The dominant term controlling the late-time approach to the constant solution is proportional to the matter density via the constant  $N$ .

#### E. Inflationary Universes

The behaviour found for lambda-dominated universes enables us to understand what would transpire during a period of de Sitter inflation during the early stages of a varying- $\alpha$  cosmology. It is straightforward to extend these conclusions to any cosmology undergoing power-law inflation. Suppose the varying- $\alpha$  Friedmann model contains a perfect fluid with  $p = (\gamma - 1)\rho$  and  $0 < \gamma < 2/3$ . The expansion scale factor will increase with  $a(t) \propto t^{2/3\gamma}$ , while  $\psi$  will be governed, to leading order by

$$(\dot{\psi}t^{2/\gamma}) = 0 \quad (50)$$

Hence, for large expansion

$$\psi = \psi_0 + Dt^{-(2-\gamma)/\gamma} \rightarrow \psi_0 \quad (51)$$

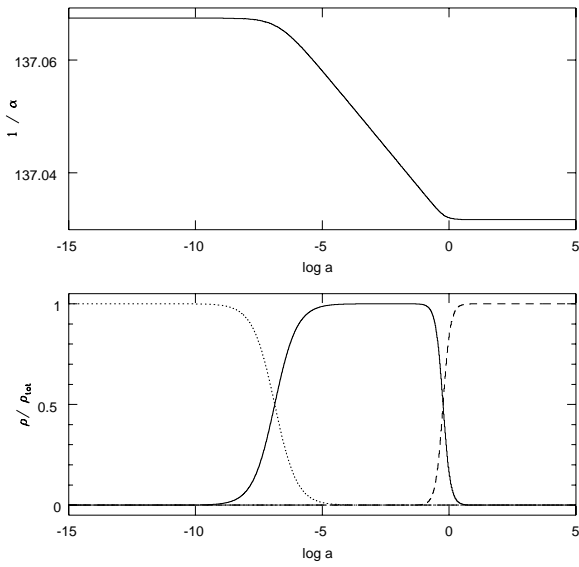


FIG. 4. The top figure shows numerical evolution of  $\alpha$  from radiation domination through matter domination and into lambda domination where the change in  $\alpha$  comes to an end. The lower plot shows radiation (dotted), matter (solid) and lambda (dashed) densities as fractions of the total energy density

and so  $\psi$  and  $\alpha$  approach a constant with power-law (exponential) rapidity during any period of power-law (de Sitter) inflation. If we evaluate the kinetic term  $O(\dot{\psi}^2)$  in the Friedmann equation and the terms  $O(N \exp[-2\psi])$  in the  $\psi$  conservation equation, we see that the assumption of  $a(t) \propto t^{2/3\gamma}$  is an increasingly good approximation as inflation proceeds. Similar behaviour would be displayed by a quintessence field which violated the strong-energy condition and came to dominate the expansion of the universe at late times. It would turn off the time variation of the fine structure constant in the same manner as the curvature of lambda terms discussed above. Note that the  $\psi$  field itself is not a possible source of inflationary behaviour in these models. We are assuming that the inflation is contributed, as usual, by some other scalar matter field with a self-interaction potential. However, if this field was charged then these conclusions could be altered as the coupling of the inflationary scalar field to the  $\psi$  field would be more complicated.

#### F. The Very Early Universe ( $t \rightarrow 0$ )

As  $t \rightarrow 0$  we expect (just as in Brans-Dicke theory) to encounter a situation where the kinetic energy of  $\psi$  dominates the evolution of  $a(t)$ . This is equivalent to the solution approaching a vacuum solution of (6)-(7) with  $\rho_m = \rho_r = 0$ , as  $t \rightarrow 0$ . In the flat case with  $\Lambda = 0$  (the  $k \neq 0$  and  $\Lambda \neq 0$  cases can be solved straightforwardly and the models with  $\rho_r \neq 0$  can also be solved exactly in

parametric form.) we have

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{4\pi G\omega}{3}\dot{\psi}^2 \quad (52)$$

$$\ddot{\psi} + 3H\dot{\psi} = 0 \quad (53)$$

Thus the exact vacuum solution is

$$\psi = \psi_0 + \frac{1}{\sqrt{12\pi G\omega}} \ln(t) \quad (54)$$

$$a = t^{1/3} \quad (55)$$

During this phase the fine structure constant *increases* as a power-law of the comoving proper time as  $t$  increases:

$$\alpha = \exp[2\psi] \propto t^{\frac{1}{\sqrt{3\pi G\omega}}} \quad (56)$$

Note that the matter and radiation density terms fall off slower than  $\dot{\psi}^2 \propto t^{-2}$  as  $t \rightarrow 0$  and  $\exp[-2\psi] \propto t^{-1/(\sqrt{3\pi G\omega})}$ . They will eventually dominate the evolution at some later time and the vacuum approximation will break down. As in Brans-Dicke cosmology [26] we expect the general solutions of the cosmological equations to approach this vacuum solution as  $t \rightarrow 0$  and to approach the other late-time asymptotes discussed above as  $t \rightarrow \infty$ .

## IV. DISCUSSION

The overall pattern of cosmological evolution is clear from the results of the last section even though it is not possible to solve the Friedmann equation exactly in most cases. There are five distinct phases:

- **a.** Near the initial singularity the kinetic part of scalar field  $\psi$  will dominate the expansion and the universe behaves like a general relativistic Friedmann universe containing a massless scalar or stiff perfect fluid field, with  $a = t^{1/3}$ . During this 'vacuum phase', the fine structure constant increases as a power law in time.
- **b.** As the universe ages the radiation density will eventually become larger than the kinetic energy of the  $\psi$  field. In this radiation dominated epoch, the fine structure constant will approach a specific solution,  $\alpha \propto t^{1/2}$  asymptotically. In reality however, if the initial value of  $\alpha$  is much larger than the specific solution, we will have a potentially very long transient period of constant evolution, and the universe may become dust dominated while  $\alpha$  is still constant.
- **c.** After dust domination begins,  $\alpha$  slowly approaches an asymptotic solution,  $\alpha = 2N \ln(t) \times \exp[-t^{-1}li(t)]$ , where  $li(t)$  is the logarithmic integral function. If the universe has zero curvature and

no cosmological constant this will approach the late time solution  $\alpha \propto \ln(t)$ .

d. If the universe is open then this increase will be brought to an end when the universe becomes dominated by spatial curvature and  $\alpha$  will approach a constant. If the curvature is positive the universe will eventually reach an expansion maximum and contract so long as there are no fluids present which violate the strong energy condition. The behaviour of closed universes also offers a good approximation to the evolution of bound spherically symmetric density inhomogeneities of large scale in a background universes and will be discussed in a separate paper.

e. If there is a positive cosmological constant, the change in  $\alpha$  will be halted when the cosmological constant starts to accelerate the universe. If any other quintessential perfect fluid with equation of state satisfying  $p < -\rho/3$  is present in the universe then it would also ultimately halt the change in  $\alpha$  when it began to dominate the expansion of the universe.

To obtain a more holistic picture of the evolution it is useful to string these different parts together. To a good approximation we know that in the vacuum phase from the Planck time  $t_p$  until  $t_v$  we have

$$a \propto t^{\frac{1}{3}}; \alpha \propto t^A; A = \frac{1}{\sqrt{3\pi G\omega}} \quad (57)$$

In the radiation era we have  $\alpha$  constant until the growth kicks in at a time  $t_{growth}$ . The fine structure constant then increases as

$$a \propto \alpha \propto t^{1/2} \quad (58)$$

until  $t_{eq}$  when the radiation era end and dust takes over. However, in universes like our own, this growth era is never reached. Then, in the dust era,

$$\alpha \propto \ln t \quad (59)$$

until the curvature or lambda eras begin at  $t_c$  or  $t_\Lambda$ , after which  $\alpha$  remains constant until the present,  $t_0$ . So, matching these phases of evolution together we can express  $\alpha(t_0)$  in terms of  $\alpha(t_p)$ :

When the universe is open with  $\Lambda = 0$ :

$$\alpha(t_0) = \alpha(t_p) \left(\frac{t_v}{t_p}\right)^A \left(\frac{t_{eq}}{t_{growth}}\right)^{1/2} \left(\frac{\ln(t_c/t_p)}{\ln(t_{eq}/t_p)}\right), \quad (60)$$

where we have used the fact that our log formula to express ages in Planck time units.

When the universe is flat with  $\Lambda > 0$ :

$$\alpha(t_0) = \alpha(t_p) \left(\frac{t_p}{t_v}\right)^A \left(\frac{t_{eq}}{t_{growth}}\right)^{1/2} \left(\frac{\ln(t_\Lambda/t_p)}{\ln(t_{eq}/t_p)}\right) \quad (61)$$

and  $t_c$  has been replaced by  $t_\Lambda$ .

For the radiation era we consider two extreme cases. We look at a constant  $\alpha$  scenario with  $t_{growth} = t_{eq}$  and a scenario where it grows throughout the radiation era,  $t_{growth} = t_v$ .

Typically,  $t_c/t_p \sim t_\Lambda/t_p \sim 10^{59}$  and  $t_{eq}/t_p \sim 10^{53}$ , so in both cases for constant  $\alpha$  evolution in the radiation epoch we get

$$\alpha(t_0) = \alpha(t_p) \left(\frac{t_v}{t_p}\right)^A \left(\frac{59}{53}\right) \sim 1.11\alpha(t_p) \left(\frac{t_v}{t_p}\right)^A \quad (62)$$

We approximate the value for  $t_v \sim t_p \sim 1$ , so for continuous growth through radiation epoch we get

$$\alpha(t_0) = \alpha(t_p) \left(\frac{t_v}{t_p}\right)^A (10^{53})^{1/2} \left(\frac{59}{53}\right) \sim 10^{26}\alpha(t_p) \left(\frac{t_v}{t_p}\right)^A \quad (63)$$

Hence there are very different possibilities for the change in  $\alpha$  depending on the evolution in the radiation era.

We have proved this sequence of phases by an exhaustive numerical and analytical study. The ensuing scenario finds two interesting applications, with which we conclude.

In [1] we found that our theory could fit simultaneously the varying  $\alpha$  results reported in [10,9,11] and the evidence for an accelerating universe presented in [27]. We noted the curious fact that there is a coincidence between the redshift at which the universe starts accelerating and the redshift where variations in  $\alpha$  have been observed but below which  $\alpha$  must stabilise to be in accord with geochemical evidence [28,29]. This may be explained dynamically in our theory by the fact that the onset of lambda domination suppresses variations in  $\alpha$ . Therefore  $\alpha$  remains almost constant in the radiation era, undergoes small logarithmic time-increase in the matter era, but approaches a constant value when the universe starts accelerating because of the presence of a positive cosmological constant. Hence, we comply with geological, nucleosynthesis, and microwave background radiation constraints on time-variations in  $\alpha$ , while fitting simultaneously the observed accelerating universe and the recent high-redshift evidence for small  $\alpha$  variations in quasar spectra.

We have also noted that within this theory the usual anthropic arguments for a lambda free universe may be reversed [30]. Usually, the anthropic principle is used to justify the near flatness and  $\Lambda \approx 0$  nature of our universe since large curvature and lambda prevents the formation of galaxies and stars from small perturbations. We have shown that it might be anthropically *disadvantageous* for a universe to lie too close to flatness or for the cosmological constant to lie too close to zero. This constraint occurs because ‘‘constants’’ change throughout the dust-dominated period when the curvature and lambda do not influence the expansion of the universe.



. The onset of a period of lambda or curvature domination has the property of dynamically stabilising the constants, thereby creating favourable conditions for the emergence of structures. If the universe were exactly flat and lambda were exactly zero then  $\alpha$  would continue to grow to a value that appears to make living complexity impossible [30].

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